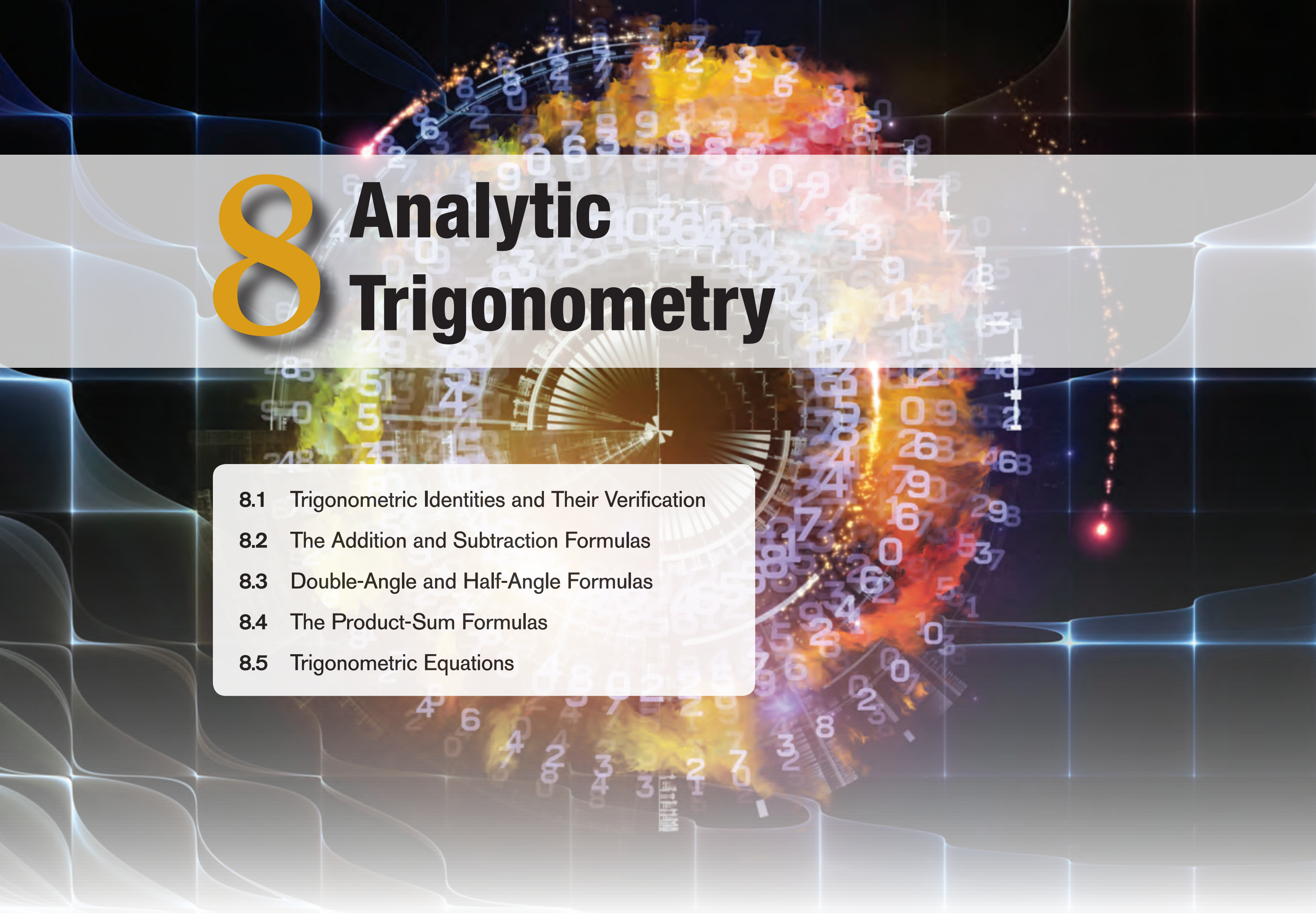

8 Analytic Trigonometry

Much of the language and terminology of algebra carries over to trigonometry. For example, we have seen that algebraic expressions involve variables, constants, and algebraic operations. **Trigonometric expressions** involve these same elements but also permit trigonometric functions of variables and constants. They also allow algebraic operations upon these trigonometric functions. Thus,

$$x + \sin x \qquad \sin x + \tan x \qquad \frac{1 - \cos x}{\sec^2 x}$$

are all examples of trigonometric expressions.

The distinction between an identity and an equation also carries over to trigonometry. Thus, a **trigonometric identity** is true for all values that may be assumed by the variable, but a **trigonometric equation** is true only for certain values of the variable, called *solutions*. (Note that the solutions of a trigonometric equation may be expressed as real numbers or as angles.) As usual, the set of all solutions of a trigonometric equation is called the *solution set*.

8.1 Trigonometric Identities and Their Verification

8.1a *Fundamental Identities*

In Section 7.4, we established the identity

$$\sin^2 t + \cos^2 t = 1 \tag{1}$$

If $\cos t \neq 0$, we may divide both sides of Equation (1) by $\cos^2 t$ to obtain

$$\frac{\sin^2 t}{\cos^2 t} + \frac{\cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$

or

$$\tan^2 t + 1 = \sec^2 t \tag{2}$$

Similarly, if $\sin t \neq 0$, dividing Equation (1) by $\sin^2 t$ yields

$$\frac{\sin^2 t}{\sin^2 t} + \frac{\cos^2 t}{\sin^2 t} = \frac{1}{\sin^2 t}$$

or

$$\cot^2 t + 1 = \csc^2 t \tag{3}$$

Observe that $\tan t$ and $\sec t$ are undefined for exactly those values of t for which $\cos t = 0$. Similarly, $\cot t$ and $\csc t$ are undefined for those values of t for which $\sin t = 0$. It follows that Equations (2) and (3) are also identities. These identities, together with those discussed in Section 7.4, are called the **fundamental identities**. We summarize them in Table 8-1.

TABLE 8-1 Fundamental Identities

$\tan t = \dfrac{\sin t}{\cos t}$	$\sin^2 t + \cos^2 t = 1$	$\tan t = \dfrac{1}{\cot t}$
$\cot t = \dfrac{\cos t}{\sin t}$	$\tan^2 t + 1 = \sec^2 t$	$\sin^2 t = 1 - \cos^2 t$
$\csc t = \dfrac{1}{\sin t}$	$\cot^2 t + 1 = \csc^2 t$	$\cos^2 t = 1 - \sin^2 t$
$\sec t = \dfrac{1}{\cos t}$	$\sin t = \dfrac{1}{\csc t}$	$\tan^2 t = \sec^2 t - 1$
$\cot t = \dfrac{1}{\tan t}$	$\cos t = \dfrac{1}{\sec t}$	$\cot^2 t = \csc^2 t - 1$

The fundamental identities can be used to simplify trigonometric expressions as well as to verify trigonometric identities. Such manipulations may enable us to see relationships that would otherwise be obscured.

The preferred method of verifying an identity is to transform one side of the equation into the other. Although we will use this method whenever practical, we recognize that it is also acceptable to transform each side independently, with the hope of arriving at the same expression and then showing that the process can be reversed. This is often the technique used when both sides of the equation involve complicated expressions.

Unfortunately, we cannot present a specific set of steps that will "always work" to transform one side into the other. In fact, there are often many ways to handle a given identity. We will, however, demonstrate different techniques that we have found to be of value. We list the following suggestions on how to proceed, with examples to follow.

1. Factoring may help to simplify an expression.
2. It is often helpful to write all of the trigonometric functions in terms of sine and cosine.
3. Consider beginning with the more complicated expression and perform some of the indicated operations.
4. If you have the ratio of two trigonometric functions, it may be worthwhile to multiply both numerator and denominator by some trigonometric expression to obtain forms such as $1 - \sin^2\theta$, $1 - \cos^2\theta$, or $\sec^2\theta - 1$, which can be further simplified.

EXAMPLE 1 *Using Trigonometric Identities*

Simplify the expression $\sin^2 x + \sin^2 x \tan^2 x$.

SOLUTION

We begin by noting that $\sin^2 x$ appears in both terms, which suggests that we factor.

$$\begin{aligned} \sin^2 x + \sin^2 x \tan^2 x &= \sin^2 x\,(1 + \tan^2 x) && \text{Factoring} \\ &= \sin^2 x \sec^2 x && 1 + \tan^2 x = \sec^2 x \\ &= \sin^2 x\left(\frac{1}{\cos^2 x}\right) && \sec x = \frac{1}{\cos x} \\ &= \tan^2 x && \frac{\sin x}{\cos x} = \tan x \end{aligned}$$

■

Progress Check

Simplify the expression $\dfrac{\csc\theta}{1 + \cot^2\theta}$.

Answer

$\sin\theta$

EXAMPLE 2 *Verifying an Identity*

Verify the identity

$$\sin\alpha - \sin^2\alpha = \frac{1 - \sin\alpha}{\csc\alpha}$$

SOLUTION

$$\begin{aligned} \sin\alpha - \sin^2\alpha &= \sin\alpha\,(1 - \sin\alpha) \\ &= \frac{1 - \sin\alpha}{\csc\alpha} \end{aligned}$$

■

Progress Check

Verify the identity

$$\frac{\sin^2 y - 1}{1 - \sin y} = -1 - \sin y$$

WARNING

When verifying identities, we are trying to show that equality holds. Since adding, subtracting, multiplying, and dividing both sides on an equation are properties that only hold for equality, we cannot use those when verifying identities.

EXAMPLE 3 *Verifying an Identity*

Verify the identity $\cos x \tan x \csc x = 1$.

SOLUTION

$$\begin{aligned}\cos x \tan x \csc x &= \cos x \left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{\sin x}\right)\\ &= 1\end{aligned}$$

■

Progress Check

Verify the identity $\sin x \sec x = \tan x$.

EXAMPLE 4 *Verifying an Identity*

Verify the identity

$$\frac{1}{1 - \sin x} + \frac{1}{1 + \sin x} = 2 \sec^2 x$$

SOLUTION

We begin with the left-hand side, combining fractions.

$$\begin{aligned}\frac{1}{1 - \sin x} + \frac{1}{1 + \sin x} &= \frac{1 + \sin x + 1 - \sin x}{(1 - \sin x)(1 + \sin x)}\\ &= \frac{2}{1 - \sin^2 x}\\ &= \frac{2}{\cos^2 x}\\ &= 2 \sec^2 x\end{aligned}$$

■

Progress Check

Verify the identity $\cos x + \tan x \sin x = \sec x$.

EXAMPLE 5 *Verifying an Identity*

Verify the identity

$$\frac{\cos\theta}{1-\sin\theta} = \sec\theta + \tan\theta$$

SOLUTION

To obtain the expression $1 - \sin^2\theta$ in the denominator, we multiply numerator and denominator by $1 + \sin\theta$.

$$\begin{aligned}
\frac{\cos\theta}{1-\sin\theta} &= \left(\frac{\cos\theta}{1-\sin\theta}\right)\left(\frac{1+\sin\theta}{1+\sin\theta}\right) \\
&= \frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta} \\
&= \frac{\cos\theta(1+\sin\theta)}{\cos^2\theta} \\
&= \frac{1+\sin\theta}{\cos\theta} \\
&= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} \\
&= \sec\theta + \tan\theta
\end{aligned}$$

■

Progress Check

Verify the identity

$$\frac{1+\cos t}{\sin t} + \frac{\sin t}{1+\cos t} = 2\csc t$$

EXAMPLE 6 *Verifying an Identity*

Verify the identity

$$\frac{\cot u - \tan u}{\sin u \cos u} = \csc^2 u - \sec^2 u$$

SOLUTION

We transform both sides of the equation by writing all trigonometric functions in terms of sine and cosine. For the left-hand side, we have

$$\begin{aligned}
\frac{\cot u - \tan u}{\sin u \cos u} &= \frac{\dfrac{\cos u}{\sin u} - \dfrac{\sin u}{\cos u}}{\sin u \cos u} \\
&= \frac{\cos^2 u - \sin^2 u}{\sin^2 u \cos^2 u}
\end{aligned}$$

and for the right-hand side, we have

$$\begin{aligned}
\csc^2 u - \sec^2 u &= \frac{1}{\sin^2 u} - \frac{1}{\cos^2 u} \\
&= \frac{\cos^2 u - \sin^2 u}{\sin^2 u \cos^2 u}
\end{aligned}$$

We have successfully transformed both sides of the equation into the same expression. Since all the steps are reversible, we have verified the identity. ■

Progress Check

Verify the identity

$$\frac{\sin x + \cos x}{\tan^2 x - 1} = \frac{\cos^2 x}{\sin x - \cos x}$$

Exercise Set 8.1

In Exercises 1–46, verify each of the identities.

1. $\csc \gamma - \cos \gamma \cot \gamma = \sin \gamma$
2. $\cot x \sec x = \csc x$
3. $\sec v + \tan v = \dfrac{1 + \sin v}{\cos v}$
4. $\cos \theta + \tan \theta \sin \theta = \sec \theta$
5. $\sin \alpha \sec \alpha = \tan \alpha$
6. $\sec \beta - \cos \beta = \sin \beta \tan \beta$
7. $3 - \sec^2 x = 2 - \tan^2 x$
8. $1 - 2 \sin^2 t = 2 \cos^2 t - 1$
9. $\dfrac{\sec^2 y}{\tan y} = \tan y + \cot y$
10. $\dfrac{\sin x + \cos x}{\cos x} = 1 + \tan x$
11. $\dfrac{\sin u}{\csc u} + \dfrac{\cos u}{\sec u} = 1$
12. $\dfrac{\tan^2 \alpha}{1 + \sec \alpha} = \sec \alpha - 1$
13. $\dfrac{\sec^2 \theta - 1}{\sec^2 \theta} = \sin^2 \theta$
14. $\sin^4 x + 2 \sin^2 x \cos^2 x + \cos^4 x = 1$
15. $\cos \gamma + \cos \gamma \tan^2 \gamma = \sec \gamma$
16. $\dfrac{1}{\tan u + \cot u} = \cos u \sin u$
17. $\dfrac{\sec w \sin w}{\tan w + \cot w} = \sin^2 w$
18. $(1 - \cos^2 \beta)(1 + \cot^2 \beta) = 1$
19. $(\sin \alpha + \cos \alpha)^2 + (\sin \alpha - \cos \alpha)^2 = 2$
20. $\dfrac{1 + \tan^2 u}{\csc^2 u} = \tan^2 u$
21. $\sec^2 v + \cos^2 v = \dfrac{\sec^4 v + 1}{\sec^2 v}$
22. $\sin^2 \theta - \tan^2 \theta = -\tan^2 \theta \sin^2 \theta$
23. $\dfrac{\sin^2 \alpha}{1 + \cos \alpha} = 1 - \cos \alpha$
24. $\cot x \sin^2 x = \cos x \sin x$
25. $\dfrac{\cos t}{1 + \sin t} = \dfrac{1 - \sin t}{\cos t}$
26. $\dfrac{\sin \beta}{1 + \cos \beta} + \dfrac{1 + \cos \beta}{\sin \beta} = 2 \csc \beta$
27. $\csc^2 \theta - \dfrac{\cos^2 \theta}{\sin^2 \theta} = 1$
28. $\dfrac{\cos^2 u}{1 - \sin u} = 1 + \sin u$
29. $\dfrac{\cot y}{1 + \cot^2 y} = \sin y \cos y$
30. $\dfrac{1 + \tan^2 x}{\tan^2 x} = \csc^2 x$
31. $\cos(-t) \csc(-t) = -\cot t$
32. $\sin(-\theta) \sec(-\theta) = -\tan \theta$
33. $\dfrac{\sec x + \csc x}{1 + \tan x} = \csc x$
34. $\dfrac{\sec u}{\sec u - 1} = \dfrac{1}{1 - \cos u}$
35. $\dfrac{1 + \tan x}{1 + \cot x} = \dfrac{\sec x}{\csc x}$
36. $(\tan u + \sec u)^2 = \dfrac{1 + \sin u}{1 - \sin u}$
37. $\dfrac{1 - \sin t}{1 + \sin t} = (\sec t - \tan t)^2$
38. $2 \csc^2 \theta - \csc^4 \theta = 1 - \cot^4 \theta$
39. $\dfrac{\sin^2 w}{\cos^4 w + \cos^2 w \sin^2 w} = \tan^2 w$
40. $\dfrac{\sin z + \tan z}{1 + \cos z} = \tan z$
41. $\dfrac{\sec \gamma - \csc \gamma}{\sec \gamma + \csc \gamma} = \dfrac{\tan \gamma - 1}{\tan \gamma + 1}$
42. $\dfrac{\cot x - 1}{1 - \tan x} = \dfrac{\csc x}{\sec x}$
43. $\dfrac{\tan \gamma - \sin \gamma}{\tan \gamma} = \dfrac{\sin^2 \gamma}{1 + \cos \gamma}$
44. $\cos^4 u - \sin^4 u = \cos^2 u - \sin^2 u$
45. $\dfrac{\csc x}{1 + \csc x} - \dfrac{\csc x}{1 - \csc x} = 2 \sec^2 x$
46. $\sin^3 \theta + \cos^3 \theta = (1 - \sin \theta \cos \theta)(\sin \theta + \cos \theta)$

In Exercises 47–52, show that each of the equations is not an identity by finding a value of the variable for which the equation is not true.

47. $\sin x = \sqrt{1 - \cos^2 x}$
48. $\tan x = \sqrt{\sec^2 x - 1}$
49. $(\sin t + \cos t)^2 = \sin^2 t + \cos^2 t$
50. $\sin \theta + \cos \theta = \sec \theta + \csc \theta$
51. $\sqrt{\cos^2 x} = \cos x$
52. $\sqrt{\cot^2 x} = \cot x$

8.2 The Addition and Subtraction Formulas

The identities that we verified in the examples and exercises of Section 8.1 were, in general, of no special significance. We were primarily interested in demonstrating manipulation with the fundamental identities. There are, however, many trigonometric identities that are indeed of importance. These identities are called **trigonometric formulas**.

Our objective in this section is to develop the **addition formulas** for $\sin(s+t)$, $\cos(s+t)$, and $\tan(s+t)$, as well as the **subtraction formulas** for $\sin(s-t)$, $\cos(s-t)$, and $\tan(s-t)$.

We begin with the derivation of $\cos(s-t)$. For convenience, assume that s, t, and $s-t$ are all positive and less than 2π. Let $P_1 = P(s)$, $Q_1 = P(t)$, and $P_2 = P(s-t)$ be the points on the unit circle determined by s, t, and $s-t$, respectively, as shown in Figure 8-1. Then $\overset{\frown}{Q_2P_2Q_1P_1} = s$, $\overset{\frown}{Q_2P_2Q_1} = t$, and $\overset{\frown}{Q_2P_2} = s-t$. From Section 7.3, we know that the coordinates of P_1, Q_1, and P_2 are $P_1 = P(\cos s, \sin s)$, $Q_1 = P(\cos t, \sin t)$, and $P_2 = P(\cos(s-t), \sin(s-t))$. Since the length of the arcs $\overset{\frown}{Q_1P_1}$ and $\overset{\frown}{Q_2P_2}$ are both equal (length of $s-t$), the length of the chords $\overline{Q_1P_1}$ and $\overline{Q_2P_2}$ are also equal. By the distance formula, we have

$$\overline{Q_1P_1} = \overline{Q_2P_2}$$

$$\sqrt{(\cos s - \cos t)^2 + (\sin s - \sin t)^2} = \sqrt{[\cos(s-t) - 1]^2 + [\sin(s-t) - 0]^2}$$

Squaring both sides and rearranging terms, we have

$$\begin{aligned}\sin^2 s + \cos^2 s + \sin^2 t + \cos^2 t - 2\cos s\cos t - 2\sin s\sin t \\ = \sin^2(s-t) + \cos^2(s-t) - 2\cos(s-t) + 1\end{aligned}$$

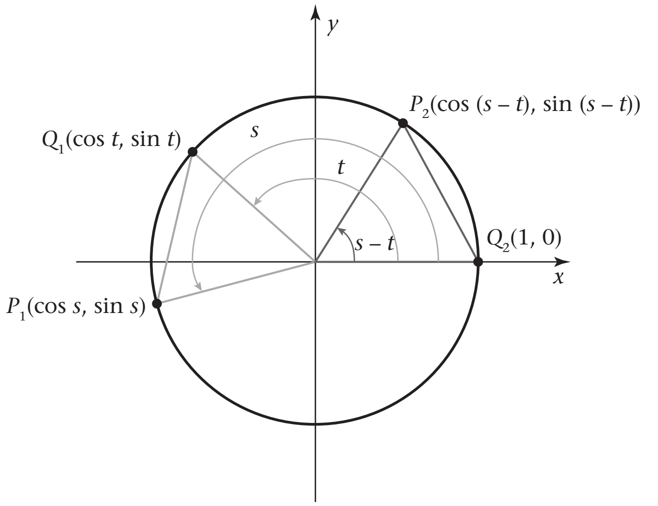

FIGURE 8-1
The Derivation of $\cos(s-t)$

Since $\sin^2 s + \cos^2 s = 1$, $\sin^2 t + \cos^2 t = 1$ and $\sin^2(s-t) + \cos^2(s-t) = 1$, we have

$$2 - 2\cos s\cos t - 2\sin s\sin t = 2 - 2\cos(s-t)$$

Solving for $\cos(s-t)$ yields the formula

$$\cos(s-t) = \cos s\cos t + \sin s\sin t \qquad (1)$$

We obtain the formula for $\cos(s+t)$ by writing

$$s + t = s - (-t)$$

Therefore

$$\begin{aligned}\cos(s+t) &= \cos(s-(-t))\\ &= \cos s\cos(-t) + \sin s\sin(-t)\end{aligned}$$

Since $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$,

$$\cos(s+t) = \cos s\cos t - \sin s\sin t \qquad (2)$$

EXAMPLE 1 *Using the Subtraction Formula*

Find $\cos 15^\circ$ without using a calculator.

SOLUTION

Since $15^\circ = 45^\circ - 30^\circ$, we may use the formula for $\cos(s-t)$ to obtain

$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ)\\ &= \cos 45^\circ\cos 30^\circ + \sin 45^\circ\sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

■

Progress Check

Redo Example 1 using $15^\circ = 60^\circ - 45^\circ$.

EXAMPLE 2 *Using the Addition Formula*

Find the exact value of $\cos\frac{5\pi}{12}$.

SOLUTION

We note that $\frac{5\pi}{12} = \frac{2\pi}{12} + \frac{3\pi}{12} = \frac{\pi}{6} + \frac{\pi}{4}$. Then

$$\begin{aligned}\cos\frac{5\pi}{12} &= \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right)\\ &= \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4}\\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

■

Progress Check

Redo Example 2 using the identity $\frac{5\pi}{12} = \frac{9\pi}{12} - \frac{4\pi}{12}$.

We may now establish the following relationships:

Cofunctions

$$\cos\left(\frac{\pi}{2}-t\right)=\sin t \qquad (3)$$

$$\sin\left(\frac{\pi}{2}-t\right)=\cos t \qquad (4)$$

$$\tan\left(\frac{\pi}{2}-t\right)=\cot t \qquad (5)$$

As we stated in Section 7.2, sine and cosine are **cofunctions**, as are tangent and cotangent, and secant and cosecant. We now verify them for all real numbers or angles here.

Using the subtraction formula for cosine, we have

$$\begin{aligned}\cos\left(\frac{\pi}{2}-t\right)&=\cos\frac{\pi}{2}\cos t+\sin\frac{\pi}{2}\sin t\\&=0\cdot\cos t+1\sin t\\&=\sin t\end{aligned}$$

which establishes the identity in Equation (3). Replacing t with $\frac{\pi}{2}-t$ in this identity yields

$$\cos\left[\frac{\pi}{2}-\left(\frac{\pi}{2}-t\right)\right]=\sin\left(\frac{\pi}{2}-t\right)$$

$$\cos t=\sin\left(\frac{\pi}{2}-t\right)$$

which establishes the identity in Equation (4). The third identity follows from the definition of tangent and from Equations (3) and (4).

$$\tan\left(\frac{\pi}{2}-t\right)=\frac{\sin\left(\frac{\pi}{2}-t\right)}{\cos\left(\frac{\pi}{2}-t\right)}=\frac{\cos t}{\sin t}=\cot t$$

We will now prove the identity in Equation (6). (We leave the proof of Equation (7) as an exercise.)

$$\sin(s+t)=\sin s\cos t+\cos s\sin t \qquad (6)$$

$$\sin(s-t)=\sin s\cos t-\cos s\sin t \qquad (7)$$

From Equation (3),

$$\begin{aligned}\sin(s+t)&=\cos\left[\frac{\pi}{2}-(s+t)\right]\\&=\cos\left[\left(\frac{\pi}{2}-s\right)-t\right]\\&=\cos\left(\frac{\pi}{2}-s\right)\cos t+\sin\left(\frac{\pi}{2}-s\right)\sin t\\&=\sin s\cos t+\cos s\sin t\end{aligned}$$

We conclude with the proof of the identity in Equation (8). (We leave the proof of Equation (9) as an exercise.)

$$\tan(s+t)=\frac{\tan s+\tan t}{1-\tan s\tan t} \qquad (8)$$

$$\tan(s-t)=\frac{\tan s-\tan t}{1+\tan s\tan t} \qquad (9)$$

$$\tan(s+t) = \frac{\sin(s+t)}{\cos(s+t)}$$
$$= \frac{\sin s\cos t + \cos s\sin t}{\cos s\cos t - \sin s\sin t}$$
$$= \frac{\left(\frac{\sin s}{\cos s}\cdot\frac{\cos t}{\cos t}\right)+\left(\frac{\cos s}{\cos s}\cdot\frac{\sin t}{\cos t}\right)}{\left(\frac{\cos s}{\cos s}\cdot\frac{\cos t}{\cos t}\right)-\left(\frac{\sin s}{\cos s}\cdot\frac{\sin t}{\cos t}\right)}$$
$$= \frac{\tan s + \tan t}{1 - \tan s\tan t}$$

EXAMPLE 3 *Applying the Addition Formula*

Show that $\sin\left(x+\frac{3\pi}{2}\right) = -\cos x$.

SOLUTION

$$\sin\left(x+\frac{3\pi}{2}\right) = \sin x\cos\frac{3\pi}{2} + \cos x\sin\frac{3\pi}{2}$$
$$= (\sin x)\,0 + (\cos x)(-1)$$
$$= -\cos x$$

■

Progress Check

Verify that $\tan(x-\pi) = \tan x$.

EXAMPLE 4 *Using the Addition Formula*

Given $\sin\alpha = -\frac{4}{5}$, with α an angle in quadrant III, and $\cos\beta = -\frac{5}{13}$, with β an angle in quadrant II, use the addition formula to find $\sin(\alpha+\beta)$ and the quadrant in which $\alpha+\beta$ lies.

SOLUTION

The addition formula

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

requires that we know $\sin\alpha$, $\cos\alpha$, $\sin\beta$, and $\cos\beta$. Using the fundamental identity $\sin^2\alpha + \cos^2\alpha = 1$, we have

$$\cos^2\alpha = 1 - \sin^2\alpha = 1 - \frac{16}{25} = \frac{9}{25}$$

Taking the square root of both sides, we must have $\cos\alpha = -\frac{3}{5}$ since α is in quadrant III. Similarly,

$$\sin^2\beta = 1 - \cos^2\beta = 1 - \frac{25}{169} = \frac{144}{169}$$

Taking the square root of both sides, we must have $\sin\beta = \frac{12}{13}$ since β is in quadrant II. Thus,

$$\sin(\alpha+\beta) = \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right)$$
$$= \frac{20}{65} - \frac{36}{65} = -\frac{16}{65}$$

Since $\sin(\alpha+\beta)$ is negative, $\alpha+\beta$ lies in either quadrant III or quadrant IV. However, the sum of an angle that lies in quadrant III and an angle that lies in quadrant II cannot lie in quadrant III. Thus, $\alpha+\beta$ lies in quadrant IV. ■

Progress Check

Given $\cos \alpha = -\frac{4}{5}$, with α in quadrant III, and $\cos \beta = \frac{3}{5}$, with β in quadrant I, find $\cos(\alpha - \beta)$ and the quadrant in which $\alpha - \beta$ lies.

Answers

$-\frac{24}{25}$, quadrant II

Exercise Set 8.2

In Exercises 1–6, show that the given equation is not an identity. (*Hint*: For each equation, find values of s and t for which the equation is not true.)

1. $\cos(s - t) = \cos s - \cos t$
2. $\sin(s + t) = \sin s + \sin t$
3. $\sin(s - t) = \sin s - \sin t$
4. $\cos(s + t) = \cos s + \cos t$
5. $\tan(s + t) = \tan s + \tan t$
6. $\tan(s - t) = \tan s - \tan t$

In Exercises 7–22, use the addition and subtraction formulas to find exact values.

7. $\cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$
8. $\sin\left(\frac{\pi}{6} - \frac{\pi}{4}\right)$
9. $\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$
10. $\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$
11. $\cos(30^\circ + 180^\circ)$
12. $\tan(60^\circ + 300^\circ)$
13. $\tan(300^\circ - 60^\circ)$
14. $\sin(270^\circ - 45^\circ)$
15. $\sin\frac{11\pi}{12}$
16. $\tan\frac{7\pi}{12}$
17. $\cos\frac{7\pi}{12}$
18. $\tan 75^\circ$
19. $\sin\frac{7\pi}{6}$
20. $\cos\frac{5\pi}{6}$
21. $\tan 15^\circ$
22. $\tan 165^\circ$

In Exercises 23–28, write the given expression in terms of cofunctions of complementary angles.

23. $\sin 47^\circ$
24. $\cos 78^\circ$
25. $\tan\frac{\pi}{6}$
26. $\tan 84^\circ$
27. $\cos\frac{\pi}{3}$
28. $\sin 72^\circ 30'$
29. If $\sin t = -\frac{3}{5}$, with t in quadrant III, find $\sin\left(\frac{\pi}{2} - t\right)$.
30. If $\cos t = -\frac{5}{13}$, with t in quadrant II, find $\sin(t - \pi)$.
31. If $\tan\theta = \frac{4}{3}$ and angle θ lies in quadrant III, find $\tan\left(\theta + \frac{\pi}{4}\right)$.
32. If $\sec\theta = \frac{5}{3}$ and angle θ lies in quadrant I, find $\sin\left(\theta + \frac{\pi}{6}\right)$.
33. If $\cos t = 0.4$, with t in quadrant IV, find $\tan(t + \pi)$.
34. If $\sec\alpha = 1.2$ and angle α lies in quadrant IV, find $\tan(\alpha - \pi)$.
35. If $\sin s = \frac{3}{5}$ and $\cos t = -\frac{12}{13}$, with s in quadrant II and t in quadrant III, find $\sin(s + t)$.
36. If $\sin s = -\frac{4}{5}$ and $\csc t = \frac{13}{5}$, with s in quadrant IV and t in quadrant II, find $\cos(s - t)$.
37. If $\cos\alpha = \frac{5}{13}$ and $\tan\beta = -2$, with angle α in quadrant I and angle β in quadrant II, find $\tan(\alpha + \beta)$.
38. If $\sec\alpha = \frac{5}{3}$ and $\cot\beta = \frac{15}{8}$, with angle α in quadrant IV and angle β in quadrant III, find $\tan(\alpha - \beta)$.

In Exercises 39–54, prove each of the following identities by transforming the left-hand side of the equation into the expression on the right-hand side.

39. $\sin 2\alpha = 2\sin\alpha\cos\alpha$
40. $\cos 2t = \cos^2 t - \sin^2 t$
41. $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$
42. $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$
43. $\cos(x - y)\cos(x + y) = \cos^2 x\cos^2 y - \sin^2 x\sin^2 y$
44. $\dfrac{\sin(s + t)}{\sin(s - t)} = \dfrac{\tan s + \tan t}{\tan s - \tan t}$
45. $\csc\left(t + \frac{\pi}{2}\right) = \sec t$
46. $\tan(\alpha + 90^\circ) = -\cot\alpha$
47. $\tan\left(x + \frac{\pi}{4}\right) = \dfrac{1 + \tan x}{1 - \tan x}$
48. $\csc(t - \pi) = -\csc t$
49. $\cot(s - t) = \dfrac{1 + \tan s\tan t}{\tan s - \tan t}$
50. $\cot(u + v) = \dfrac{\cot u\cot v - 1}{\cot u + \cot v}$
51. $\sin(s + t) + \sin(s - t) = 2\sin s\cos t$
52. $\cos(s + t) + \cos(s - t) = 2\cos s\cos t$
53. $\dfrac{\sin(x + h) - \sin x}{h} = \sin x\left(\dfrac{\cos h - 1}{h}\right) + \cos x\left(\dfrac{\sin h}{h}\right)$
54. $\dfrac{\cos(x + h) - \cos x}{h} = \cos x\left(\dfrac{\cos h - 1}{h}\right) - \sin x\left(\dfrac{\sin h}{h}\right)$

8.3 Double-Angle and Half-Angle Formulas

8.3a *Double-Angle Formulas*

Our initial objective in this section is to derive expressions for $\sin 2t$, $\cos 2t$, and $\tan 2t$ in terms of trigonometric functions of t. We will establish the following **double-angle formulas**.

$$\sin 2t = 2\sin t\cos t \qquad (1)$$

$$\cos 2t = \cos^2 t - \sin^2 t \qquad (2)$$

$$\tan 2t = \frac{2\tan t}{1-\tan^2 t} \qquad (3)$$

To establish Equation (1), we rewrite $2t$ as $(t+t)$ and use the addition formula.

$$\begin{aligned}\sin 2t &= \sin(t+t)\\ &= \sin t\cos t + \cos t\sin t\\ &= 2\sin t\cos t\end{aligned}$$

We proceed in the same manner to prove Equation (2).

$$\begin{aligned}\cos 2t &= \cos(t+t)\\ &= \cos t\cos t - \sin t\sin t\\ &= \cos^2 t - \sin^2 t\end{aligned}$$

Using the addition formula for the tangent function yields a proof of Equation (3).

$$\begin{aligned}\tan 2t &= \tan(t+t)\\ &= \frac{\tan t+\tan t}{1-\tan t\tan t}\\ &= \frac{2\tan t}{1-\tan^2 t}\end{aligned}$$

EXAMPLE 1 *Using the Double-Angle Formulas*

If $\cos t = -\frac{3}{5}$ and $P(t)$ is in quadrant II, evaluate $\sin 2t$ and $\cos 2t$. In which quadrant does $P(2t)$ lie?

SOLUTION

We first find $\sin t$ by use of the fundamental identity $\sin^2 t + \cos^2 t = 1$. Thus,

$$\sin^2 t + \frac{9}{25} = 1$$

$$\sin^2 t = \frac{16}{25}$$

Since $P(t)$ is in quadrant II, $\sin t$ must be positive. Therefore,

$$\sin t = \frac{4}{5}$$

Applying the double-angle formulas with $\cos t = -\frac{3}{5}$ and $\sin t = \frac{4}{5}$, we have

$$\sin 2t = 2\sin t\cos t = 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$$

$$\cos 2t = \cos^2 t - \sin^2 t = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

Since $\sin 2t$ and $\cos 2t$ are both negative, we conclude that $P(2t)$ lies in quadrant III. ■

Progress Check

If $\sin\theta = \frac{5}{13}$ and θ is in quadrant I, evaluate $\sin 2\theta$ and $\tan 2\theta$.

Answers

$\sin 2\theta = \frac{120}{169}$, $\tan 2\theta = \frac{120}{119}$

EXAMPLE 2 *Using the Addition and Double-Angle Formulas*

Express $\sin 3t$ in terms of $\sin t$ and $\cos t$.

SOLUTION

We write $3t$ as $(2t + t)$. Then

$$\begin{aligned} \sin 3t &= \sin(2t + t) \\ &= \sin 2t \cos t + \cos 2t \sin t \\ &= 2 \sin t \cos t \cos t + (\cos^2 t - \sin^2 t) \sin t \\ &= 2 \sin t \cos^2 t + \sin t \cos^2 t - \sin^3 t \\ &= 3 \sin t \cos^2 t - \sin^3 t \end{aligned}$$

■

Progress Check

Express $\cos 3t$ in terms of $\sin t$ and $\cos t$.

Answer

$\cos 3t = \cos^3 t - 3 \sin^2 t \cos t$

If we begin with the formula for $\cos 2t$ and use the fundamental identity $\cos^2 t = 1 - \sin^2 t$, we obtain

$$\begin{aligned} \cos 2t &= \cos^2 t - \sin^2 t \\ &= (1 - \sin^2 t) - \sin^2 t \\ &= 1 - 2 \sin^2 t \end{aligned}$$

Similarly, replacing $\sin^2 t$ by $1 - \cos^2 t$ yields

$$\begin{aligned} \cos 2t &= \cos^2 t - \sin^2 t \\ &= \cos^2 t - (1 - \cos^2 t) \\ &= 2 \cos^2 t - 1 \end{aligned}$$

We then have two additional formulas for $\cos 2t$.

$$\cos 2t = 1 - 2 \sin^2 t \qquad (4)$$

$$\cos 2t = 2 \cos^2 t - 1 \qquad (5)$$

EXAMPLE 3 *Using Double-Angle Formulas in Verifying an Identity*

Verify the identity

$$\frac{1 - \cos 2\alpha}{2 \sin \alpha \cos \alpha} = \tan \alpha$$

SOLUTION

Substituting $\cos 2\alpha = 1 - 2 \sin^2 \alpha$, we have

$$\begin{aligned} \frac{1 - \cos 2\alpha}{2 \sin \alpha \cos \alpha} &= \frac{1 - (1 - 2\sin^2 \alpha)}{2 \sin \alpha \cos \alpha} \\ &= \frac{2 \sin^2 \alpha}{2 \sin \alpha \cos \alpha} \\ &= \frac{\sin \alpha}{\cos \alpha} \\ &= \tan \alpha \end{aligned}$$

■

Progress Check

Verify the identity

$$\frac{1 + \cos 2\theta}{\sin 2\theta} = \cot \theta$$

WARNING

Note that

$$\frac{\sin 2t}{2} \neq \sin t$$

From Equation (1),

$$\frac{\sin 2t}{2} = \frac{2 \sin t \cos t}{2} = \sin t \cos t$$

8.3b *Half-Angle Formulas*

If we begin with the alternative forms for $\cos 2t$ given in Equations (4) and (5), we can obtain the following expressions for $\sin^2 t$ and $\cos^2 t$. The expressions are often used in calculus.

$$\sin^2 t = \frac{1 - \cos 2t}{2} \qquad (6)$$

$$\cos^2 t = \frac{1 + \cos 2t}{2} \qquad (7)$$

We will use the identities in Equations (6) and (7) to derive formulas for $\sin \frac{t}{2}$, $\cos \frac{t}{2}$, and $\tan \frac{t}{2}$. Substituting $s = 2t$ into Equations (6) and (7), we obtain

$$\sin^2 \frac{s}{2} = \frac{1 - \cos s}{2}$$

$$\cos^2 \frac{s}{2} = \frac{1 - \cos s}{2}$$

Replacing s with t and solving, we have

$$\sin\frac{t}{2} = \pm\sqrt{\frac{1-\cos t}{2}} \tag{8}$$

$$\cos\frac{t}{2} = \pm\sqrt{\frac{1+\cos t}{2}} \tag{9}$$

The appropriate sign to use in Equations (8) and (9) depends on the quadrant in which $P\left(\frac{t}{2}\right)$ is located. Thus, $\sin\frac{t}{2}$ is positive if $P\left(\frac{t}{2}\right)$ lies in quadrant I or II. Similarly, we choose the positive root for $\cos\frac{t}{2}$ in Equation (9) if $P\left(\frac{t}{2}\right)$ lies in quadrant I or IV.

Using the identity

$$\tan\frac{t}{2} = \frac{\sin\frac{t}{2}}{\cos\frac{t}{2}}$$

we obtain

$$\tan\frac{t}{2} = \pm\sqrt{\frac{1-\cos t}{1+\cos t}} \tag{10}$$

Formulas (8), (9), and (10) are known as the **half-angle formulas**.

EXAMPLE 4 *Applying the Half-Angle Formulas*

Find the exact values of $\sin 22.5^\circ$ and $\cos 112.5^\circ$.

SOLUTION

Applying the half-angle formulas with $22.5^\circ = \frac{45^\circ}{2}$, we have

$$\begin{aligned}\sin 22.5^\circ &= \sin\frac{45^\circ}{2}\\ &= \sqrt{\frac{1-\cos 45^\circ}{2}} = -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}\\ &= \sqrt{\frac{2-\sqrt{2}}{2}}\end{aligned}$$

Note that we chose the positive square root since 22.5° is in the first quadrant and the sine function is positive in the first quadrant. Similarly,

$$\begin{aligned}\cos 112.5^\circ &= \cos\frac{225^\circ}{2}\\ &= -\sqrt{\frac{1+\cos 225^\circ}{2}}\\ &= -\sqrt{\frac{1-\cos 45^\circ}{2}} = -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}\\ &= -\sqrt{\frac{2-\sqrt{2}}{2}}\end{aligned}$$

The negative square root was selected since 112.5° is in the second quadrant and the cosine function is negative in quadrant II. ■

Progress Check

Use the half-angle formulas to evaluate $\tan \frac{3\pi}{8}$.

Answer

$\sqrt{2} + 1$

EXAMPLE 5 *Applying the Half-Angle Formulas*

If $\sin\theta = -\frac{3}{5}$ and θ is in quadrant III, evaluate $\cos\frac{\theta}{2}$.

SOLUTION

We first evaluate $\cos\theta$ by using the identity

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \frac{9}{25} = \frac{16}{25}$$

Since θ is in quadrant III, $\cos\theta$ is negative. Thus, $\cos\theta = -\frac{4}{5}$. Note that since $180^\circ < \theta < 270^\circ$, we see that $90^\circ < \frac{\theta}{2} < 135^\circ$. Thus, $\frac{\theta}{2}$ is in quadrant II and $\cos\frac{\theta}{2}$ is negative. We can now employ the half-angle formula

$$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}}$$

$$= -\sqrt{\frac{1-\frac{4}{5}}{2}}$$

$$= -\frac{\sqrt{10}}{10}$$

■

Progress Check

If $\tan\alpha = \frac{3}{4}$ and α is in quadrant III, evaluate $\tan\frac{\alpha}{2}$.

Answer

-3

Exercise Set 8.3

In Exercises 1–12, use the given conditions to determine the value of the specified trigonometric function.

1. If $\sin u = \frac{3}{5}$ and $P(u)$ is in quadrant II, find $\cos 2u$.
2. If $\cos x = -\frac{5}{13}$ and $P(x)$ is in quadrant III, find $\sin 2x$.
3. If $\sec \alpha = -2$ and α is in quadrant II, find $\sin 2\alpha$.
4. If $\tan \theta = \frac{4}{3}$ and θ is in quadrant I, find $\cos 2\theta$.
5. If $\csc t = -\frac{17}{8}$ and $P(t)$ is in quadrant IV, find $\tan 2t$.
6. If $\cot \beta = \frac{3}{4}$ and β is in quadrant III, find $\cot 2\beta$.
7. If $\sin 2\alpha = -\frac{4}{5}$ and 2α is in quadrant IV, find $\sin 4\alpha$.
8. If $\sec 5x = -\frac{13}{12}$ and $P(5x)$ is in quadrant III, find $\tan 10x$.
9. If $\cos \frac{\theta}{2} = \frac{8}{17}$ and $\frac{\theta}{2}$ is acute, find $\cos \theta$.
10. If $\csc \frac{t}{2} = -\frac{13}{5}$ and $P\left(\frac{t}{2}\right)$ is in quadrant IV, find $\cos t$.
11. If $\sin 42^\circ \approx 0.67$, find $\cos 84^\circ$.
12. If $\cos 77^\circ \approx 0.22$, find $\cos 154^\circ$.

In Exercises 13–18, use the half-angle formulas to find exact values for each of the following.

13. $\sin 15^\circ$
14. $\cos 75^\circ$
15. $\tan \frac{\pi}{8}$
16. $\sec \frac{5\pi}{8}$
17. $\csc 165^\circ$
18. $\cot \frac{7\pi}{12}$

In Exercises 19–26, use the given conditions to determine the exact value of the specified trigonometric function.

19. If $\sin \theta = -\frac{4}{5}$ and θ is in quadrant IV, find $\cos \frac{\theta}{2}$.
20. If $\cos \theta = \frac{3}{5}$ and θ is in quadrant I, find $\sin \frac{\theta}{2}$.
21. If $\sec t = -3$ and $P(t)$ is in quadrant II, find $\sin \frac{t}{2}$.
22. If $\tan x = \frac{4}{3}$ and $P(x)$ is in quadrant III, find $\cos \frac{x}{2}$.
23. If $\cot \beta = \frac{3}{4}$ and β is in quadrant III, find $\tan \frac{\beta}{2}$.
24. If $\csc \alpha = \frac{13}{5}$ and α is in quadrant II, find $\tan \frac{\alpha}{2}$.
25. If $\cos 4x = \frac{1}{3}$ and $P(4x)$ is in quadrant IV, find $\cos 2x$.
26. If $\sec 6\alpha = -\frac{13}{12}$ and 6α is in quadrant III, find $\sin 3\alpha$.

In Exercises 27–46, verify the identity.

27. $\sin 50x = 2 \sin 25x \cos 25x$
28. $(\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$
29. $\tan 2y = \dfrac{2\cot y}{\csc^2 y - 2}$
30. $2 \sin^2 2t + \cos 4t = 1$
31. $\sin 4\alpha = 4 \sin \alpha \cos^3 \alpha - 4 \sin^3 \alpha \cos \alpha$
32. $\cos 4\beta = 1 - 8 \sin^2 \beta \cos^2 \beta$
33. $\cos 2u = \dfrac{1 - \tan^2 u}{1 + \tan^2 u}$
34. $\sin 2\theta = \dfrac{2\tan \theta}{1 + \tan^2 \theta}$
35. $\sin \dfrac{t}{2} \cos \dfrac{t}{2} = \dfrac{\sin t}{2}$
36. $\tan \dfrac{y}{2} = \csc y - \cot y$
37. $\sin \alpha - \cos \alpha \tan \dfrac{\alpha}{2} = \tan \dfrac{\alpha}{2}$
38. $\dfrac{1 - \cos 2\beta}{1 + \cos 2\beta} = \tan^2 \beta$
39. $\cos^4 x - \sin^4 x = \cos 2x$
40. $\dfrac{\sin 2t}{\sin t} - \dfrac{\cos 2t}{\cos t} = \sec t$
41. $\dfrac{2\tan \alpha}{1 + \tan^2 \alpha} = \sin 2\alpha$
42. $\cos^2 \dfrac{x}{2} = \dfrac{\tan x + \sin x}{2\tan x}$
43. $\sec 2t = \dfrac{\sec^2 t}{2 - \sec^2 t}$
44. $\cos 2t + \cot 2t = \cot 2t (\sin t + \cos t)^2$
45. $\tan \dfrac{t}{2} = \dfrac{1 - \cos t}{\sin t}$
46. $\tan \dfrac{t}{2} = \dfrac{\sin t}{1 + \cos t}$

In Exercises 47–50, find the requested value.

47. $\sin\left(2\arccos \frac{3}{5}\right)$
48. $\cos\left(2\sin^{-1} \frac{3}{5}\right)$
49. $\tan\left(2\arcsin \frac{5}{13}\right)$
50. $\cos\left(2\arctan \frac{12}{5}\right)$

8.4 The Product-Sum Formulas

The objective of this section is to derive formulas that can transform sums of sines and cosines into products of sines and cosines, and vice versa. We use the word "sum" in a more general way to include the word "difference," since subtraction can be thought of as adding a negative quantity.

The following formulas express a product as a sum.

Product-Sum Formulas

$$\sin s \cos t = \frac{\sin(s+t)+\sin(s-t)}{2} \qquad (1)$$

$$\cos s \sin t = \frac{\sin(s+t)-\sin(s-t)}{2} \qquad (2)$$

$$\cos s \cos t = \frac{\cos(s+t)+\cos(s-t)}{2} \qquad (3)$$

$$\sin s \sin t = \frac{\cos(s-t)-\cos(s+t)}{2} \qquad (4)$$

To prove the identity in Equation (1), we begin with the right-hand side of the equation.

$$\begin{aligned}\frac{\sin(s+t)+\sin(s-t)}{2} &= \frac{(\sin s\cos t+\cos s\sin t)+(\sin s\cos t-\cos s\sin t)}{2}\\ &= \frac{2\sin s\cos t}{2}\\ &= \sin s \cos t\end{aligned}$$

The proofs of the identities in Equations (2), (3), and (4) are very similar.

EXAMPLE 1 *Applying the Product-Sum Formulas*

Express $\sin 4x \cos 3x$ as a sum or a difference.

SOLUTION

Applying Equation (1), we obtain

$$\begin{aligned}\sin 4x \cos x &= \frac{\sin(4x+3x)+\sin(4x-3x)}{2}\\ &= \frac{\sin 7x+\sin x}{2}\end{aligned}$$

■

Progress Check

Express $\sin 5x \sin 2x$ as a sum or as a difference.

Answer

$\frac{1}{2}(\cos 3x-\cos 7x)$

EXAMPLE 2 *Applying the Product-Sum Formulas*

Evaluate the product $\cos\frac{5\pi}{8}\cos\frac{3\pi}{8}$ by using a product-sum formula.

SOLUTION

Using Equation (3), we have

$$\begin{aligned}\cos\frac{5\pi}{8}\cos\frac{3\pi}{8} &= \frac{1}{2}\left[\cos\left(\frac{5\pi}{8}+\frac{3\pi}{8}\right)+\cos\left(\frac{5\pi}{8}-\frac{3\pi}{8}\right)\right]\\ &= \frac{1}{2}\left[\cos\pi+\cos\frac{\pi}{4}\right]\\ &= \frac{1}{2}\left[-1+\frac{\sqrt{2}}{2}\right]=\frac{\sqrt{2}-2}{4}\end{aligned}$$

Progress Check

Evaluate $\cos\frac{\pi}{3}\sin\frac{\pi}{6}$ by a product-sum formula.

Answer

$\frac{1}{4}$

The following formulas express a sum as a product.

Sum-Product Formulas

$$\sin s+\sin t = 2\sin\frac{s+t}{2}\cos\frac{s-t}{2} \quad (5)$$

$$\sin s-\sin t = 2\cos\frac{s+t}{2}\sin\frac{s-t}{2} \quad (6)$$

$$\cos s+\cos t = 2\cos\frac{s+t}{2}\cos\frac{s-t}{2} \quad (7)$$

$$\cos s-\cos t = -2\sin\frac{s+t}{2}\sin\frac{s-t}{2} \quad (8)$$

To prove the identity in Equation (5), we begin with the right-hand side and apply Equation (1). Then

$$\begin{aligned}2\sin\frac{s+t}{2}\cos\frac{s-t}{2} &= 2\left\{\frac{1}{2}\left[\sin\left(\frac{s+t}{2}+\frac{s-t}{2}\right)+\sin\left(\frac{s+t}{2}-\frac{s-t}{2}\right)\right]\right\}\\ &= \sin s+\sin t\end{aligned}$$

This establishes Equation (5). The other identities are established in a similar fashion.

EXAMPLE 3 *Applying the Product-Sum Formulas*

Express $\sin 5x - \sin 3x$ as a product.

SOLUTION

Using Equation (6), we have

$$\begin{aligned}\sin 5x - \sin 3x &= 2\cos\frac{5x+3x}{2}\sin\frac{5x-3x}{2}\\ &= 2\cos 4x \sin x\end{aligned}$$

■

Progress Check

Express $\cos 6x + \cos 2x$ as a product.

Answer

$2\cos 4x \cos 2x$

EXAMPLE 4 *Applying the Product-Sum Formulas*

Evaluate $\cos\frac{5\pi}{12} - \cos\frac{\pi}{12}$ using a product-sum-formula.

SOLUTION

Using Equation (8), we have

$$\begin{aligned}\cos\frac{5\pi}{12} - \cos\frac{\pi}{12} &= -2\sin\frac{\pi}{4}\sin\frac{\pi}{6}\\ &= -2\left(\frac{\sqrt{2}}{2}\right)\frac{1}{2} = -\frac{\sqrt{2}}{2}\end{aligned}$$

■

Progress Check

Evaluate

$$\sin\frac{11\pi}{12} - \sin\frac{5\pi}{12}$$

using a product-sum formula.

Answer

$-\frac{\sqrt{2}}{2}$

Exercise Set 8.4

In Exercises 1–8, express each product as a sum or difference.

1. $2 \sin 5\alpha \cos \alpha$
2. $-3 \cos 6x \sin 2x$
3. $\sin 3x \sin(-2x)$
4. $\cos 7t \cos(-3t)$
5. $-2 \cos 2\theta \cos 5\theta$
6. $\sin \frac{5\theta}{2} \sin \frac{\theta}{2}$
7. $\cos(\alpha+\beta) \cos(\alpha-\beta)$
8. $-\sin 2u \cos 4u$

In Exercises 9–12, evaluate each product by using a product-sum formula.

9. $\cos \frac{7\pi}{8} \sin \frac{5\pi}{8}$
10. $\cos \frac{\pi}{3} \cos \frac{\pi}{6}$
11. $\sin 120^\circ \cos 60^\circ$
12. $\sin \frac{13\pi}{12} \sin \frac{11\pi}{12}$

In Exercises 13–20, express each sum or difference as a product.

13. $\sin 5x + \sin x$
14. $\cos 8t - \cos 2t$
15. $\cos 2\theta + \cos 6\theta$
16. $\sin 5\alpha - \sin 7\alpha$
17. $\sin(\alpha+\beta) + \sin(\alpha-\beta)$
18. $\cos \frac{x}{2} - \cos \frac{3x}{2}$
19. $\sin 7x - \sin 3x$
20. $\cos 5\theta + \cos 3\theta$

In Exercises 21–24, evaluate each sum by using a product-sum formula.

21. $\cos 75^\circ + \cos 15^\circ$
22. $\sin \frac{5\pi}{12} + \sin \frac{\pi}{12}$
23. $\cos \frac{3\pi}{4} - \cos \frac{\pi}{4}$
24. $\sin \frac{13\pi}{12} - \sin \frac{5\pi}{12}$

In Exercises 25–34, verify the identities.

25. $\sin 40^\circ + \sin 20^\circ = \cos 10^\circ$
26. $\cos 70^\circ - \cos 10^\circ = -\sin 40^\circ$
27. $\dfrac{\sin 5\theta - \sin 3\theta}{\cos 3\theta - \cos 5\theta} = \cot 4\theta$
28. $\dfrac{\cos 7x - \cos x}{\sin 7x + \sin x} = -\tan 3x$
29. $\dfrac{\sin t - \sin s}{\cos t - \cos s} = -\cot \dfrac{s+t}{2}$
30. $\dfrac{\sin s + \sin t}{\cos s + \cos t} = \tan \dfrac{s+t}{2}$
31. $\dfrac{\sin 50^\circ - \sin 10^\circ}{\cos 50^\circ - \cos 10^\circ} = -\sqrt{3}$
32. $2 \sin\left(\theta + \frac{\pi}{4}\right) \sin\left(\theta - \frac{\pi}{4}\right) = -\cos 2\theta$
33. $\dfrac{\cot x - \tan x}{\cos x + \tan x} = \cos 2x$
34. $\cos 6x \cos 2x + \sin^2 4x = \cos^2 2x$
35. Express $\sin ax \cos bx$ as a sum.
36. Express $\cos ax \cos bx$ as a sum.
37. Prove the product-sum formulas given in Equations (2), (3), and (4).
38. Prove the product-sum formulas given in Equations (6), (7), and (8).

8.5 Trigonometric Equations

Thus far, this chapter has dealt exclusively with trigonometric identities. We now seek to solve trigonometric equations that may be true for some values of the variable but not for all values.

We have seen that algebraic equations may have just one or two solutions. The situation is quite different with trigonometric equations. Since trigonometric functions are periodic by nature, if we find one solution, there must be an infinite number of solutions. To deal with this situation, we first seek all solutions t such that $0 \leq t < 2\pi$. Then, for every integer n, $t + 2\pi n$ is also a solution. The following example illustrates this procedure for finding the solution set.

EXAMPLE 1 *Solving a Trigonometric Equation*

Find all solutions of the equation $\cos t = 0$.

SOLUTION

The only values in the interval $[0, 2\pi)$ for which $\cos t = 0$ are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Then every solution is included among those values of t such that

$$t = \frac{\pi}{2} + 2\pi n \qquad \text{or} \qquad t = \frac{3\pi}{2} + 2\pi n, \qquad n \text{ an integer}$$

Since $\frac{3\pi}{2} = \frac{\pi}{2} + \pi$, the solution set can be written in a more compact form as

$$t = \frac{\pi}{2} + \pi n, \qquad n \text{ an integer}$$

■

Factoring provides an important technique for solving trigonometric equations. If we can write the equation in the form $P(x)Q(x) = 0$, we can find the solutions by setting $P(x) = 0$ and $Q(x) = 0$.

It may also be helpful to think of substituting a new variable for some trigonometric expression. Thus, the equation

$$4 \sin^2 x + 3 \sin x - 1 = 0$$

can be viewed as a quadratic in u

$$4u^2 + 3u - 1 = 0$$

by substituting $u = \sin x$.

EXAMPLE 2 *Restricting the Solutions of a Trigonometric Equation*

Find all solutions of the equation $2 \cos^2 t - \cos t - 1 = 0$ in the interval $[0, 2\pi)$.

SOLUTION

Factoring the left side of the equation yields

$$(2 \cos t + 1)(\cos t - 1) = 0$$

Setting each factor equal to 0, we have

$$2 \cos t + 1 = 0 \qquad \text{or} \qquad \cos t - 1 = 0$$

so that

$$\cos t = -\frac{1}{2} \quad \text{or} \qquad \cos t = 1$$

The solutions of $\cos t = -\frac{1}{2}$ in the interval $[0, 2\pi)$ are $t = \frac{2\pi}{3}$ and $t = \frac{4\pi}{3}$. The only solution of $\cos t = 1$ in the interval $[0, 2\pi)$ is $t = 0$. Thus, the solutions of the original problem are

$$t = \frac{2\pi}{3}, \qquad t = \frac{4\pi}{3}, \qquad \text{and} \qquad t = 0$$ ■

Progress Check

Find all solutions of the equation $2\sin^2 t - 3\sin t + 1 = 0$ in the interval $[0, 2\pi)$.

Answers

$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}$

If the solutions of the trigonometric equation are angles, the answer may be given in either radians or degrees.

EXAMPLE 3 *Expressing Solutions in Radians and Degrees*

Find all solutions of the equation $\tan\theta\cos^2\theta - \tan\theta = 0$.

SOLUTION

Factoring the left side yields

$$(\tan\theta)(\cos^2\theta - 1) = 0$$

Setting each factor equal to 0,

$$\tan\theta = 0 \qquad \text{or} \qquad \cos^2\theta = 1$$

so that

$$\tan\theta = 0, \qquad \cos\theta = 1, \qquad \text{or} \qquad \cos\theta = -1$$

These equations yield the following solutions in the interval $[0, 2\pi)$.

$$\begin{aligned} \tan\theta = 0: &\quad \theta = 0 \quad \text{or} \quad \theta = \pi \\ \cos\theta = 1: &\quad \theta = 0 \\ \cos\theta = -1: &\quad \theta = \pi \end{aligned}$$

The solutions of the original equation are

$$\theta = 0 + 2\pi n \qquad \text{and} \qquad \theta = \pi + 2\pi n, \qquad n \text{ an integer}$$

which can be expressed more compactly as

$$\theta = \pi n, \qquad n \text{ an integer}$$

In degree measure, the solutions are

$$\theta = 180^\circ n, \qquad n \text{ an integer}$$ ■

EXAMPLE 4 *Expressing Restricted Solutions in Radians and Degrees*

Find all solutions of the equation $\sin 2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi)$ and when $0^\circ \le \theta < 360^\circ$.

SOLUTION

Using the identity $\sin 2\theta = 2\sin\theta\cos\theta$, we have

$$2\sin\theta\cos\theta - 3\sin\theta = 0$$
$$\sin\theta\,(2\cos\theta - 3) = 0$$
$$\sin\theta = 0 \quad \text{or} \quad 2\cos\theta - 3 = 0$$
$$\sin\theta = 0 \quad \text{or} \quad \cos\theta = \frac{3}{2}$$

The equation $\cos\theta = \frac{3}{2}$ has no solutions. The solutions of $\sin\theta = 0$ are $\theta = 0$ and $\theta = \pi$. Therefore, the solutions of the original equation are

$$\theta = 0 \quad \text{and} \quad \theta = \pi.$$

Equivalently, in degree measure, the solutions are

$$\theta = 0^\circ \quad \text{and} \quad \theta = 180^\circ.$$

■

Progress Check

Find all solutions of the equation $\cos 2\theta + \cos\theta = 0$ in both radians and degrees.

Answers

$\frac{\pi}{3} + 2\pi n,\ \pi + 2\pi n,\ \frac{5\pi}{3} + 2\pi n$ or $60^\circ + 360^\circ n,\ 180^\circ + 360^\circ n,\ 300^\circ + 360^\circ n$

Equations involving multiple angles can often be solved by using a substitution of variable. The following example shows what may occur when seeking solutions in the interval $[0, 2\pi)$.

EXAMPLE 5 *Substitution of a Variable in Trigonometric Equations*

Find all solutions of the equation $\cos 3x = 0$ in the interval $[0, 2\pi)$.

SOLUTION

We are given

$$\cos 3x = 0, \qquad 0 \le x < 2\pi$$

Substituting $t = 3x$, we obtain

$$\cos t = 0, \qquad 0 \le \frac{t}{3} < 2\pi$$

or

$$\cos t = 0, \qquad 0 \le t < 6\pi$$

Note that we seek solutions of $\cos t = 0$ in the interval $[0, 6\pi)$ rather than $[0, 2\pi)$. The solutions are then

$$t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$$

Since $x = \frac{t}{3}$, we obtain

$$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$$

WARNING

When you perform a substitution of variable, you must remember to go back and express the answers in terms of the original variable.

EXAMPLE 6 *Substitution of Variable*

Find all solutions of the equation

$$3 \tan^2 x + \tan x - 1 = 0$$

in the interval $[0, \pi)$.

SOLUTION

Since the equation does not factor easily, consider $u = \tan x$. Then we obtain

$$3u^2 + u - 1 = 0$$

From the quadratic formula,

$$u = \frac{-1 \pm \sqrt{13}}{6}$$

so that

$$\tan x = \frac{-1 \pm \sqrt{13}}{6}$$

Therefore,

$$\tan x \approx 0.4342586 \quad \text{and} \quad \tan x \approx -0.7675919$$

in which case

$$x \approx \tan^{-1} 0.4342586 \approx 0.4096865$$

and

$$x \approx \tan^{-1}(-0.7675919) \approx -0.6546652$$

Although the first value for x is in the interval $[0, \pi)$, the second value for x is not. In fact, this second value is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the range of $\tan^{-1} x$. Since the period of $\tan x$ is π,

$$\begin{aligned} x &\approx -0.6546652 + \pi \\ &\approx 2.4869275 \end{aligned}$$

is also a solution. Observe that this new value for x is in the interval $[0, \pi)$. Thus the solutions in the given interval are $x \approx 0.4097$ and $x \approx 2.4869$.

Exercise Set 8.5

In Exercises 1–20, find all solutions of the given equation in the interval $[0, 2\pi)$ and on $[0^\circ, 360^\circ)$.

1. $2 \sin \theta - 1 = 0$
2. $2 \cos \theta + 1 = 0$
3. $\cos \alpha + 1 = 0$
4. $\cot \gamma + 1 = 0$
5. $4 \cos^2 \alpha = 3$
6. $\tan^2 \theta = 3$
7. $3 \tan^2 \alpha = 1$
8. $2 \cos^2 \alpha - 1 = 0$
9. $2 \sin^2 \beta = \sin \beta$
10. $\sin \alpha = \cos \alpha$
11. $2 \cos^2 \theta - 3 \cos \theta + 1 = 0$
12. $2 \sin^2 \theta - \sin \theta - 1 = 0$
13. $\sin 5\theta = 1$
14. $\tan 3\beta = -\sqrt{3}$
15. $2 \sin^2 \alpha - 3 \cos \alpha = 0$
16. $2 \cos^2 \theta - 1 = \sin \theta$
17. $\csc 2\theta = 2$
18. $\cos^2 2\alpha = \frac{1}{4}$
19. $\sin^2 \beta + 3 \cos \beta - 3 = 0$
20. $2 \cos^2 \theta \tan \theta - \tan \theta = 0$

In Exercises 21–38, find all the solutions of the given equation.

21. $3 \tan^2 x - 1 = 0$
22. $2 \sin^2 y - 1 = 0$
23. $3 \cot^2 \theta - 1 = 0$
24. $1 - 4 \cos^2 t = 0$
25. $\sec 2u - 2 = 0$
26. $\tan 3x - 1 = 0$
27. $\sin 4x = 0$
28. $\cos 5t = -1$
29. $4 \cos^2 2t - 3 = 0$
30. $\csc^2 2x - 2 = 0$
31. $\sin 2t + 2 \cos t = 0$
32. $\sin 2t + 3 \cos t = 0$
33. $\cos 2t + \sin t = 0$
34. $2 \cos 2t + 2 \sin t = 0$
35. $\tan^2 x - \tan x = 0$
36. $\sec^2 x - 3 \sec x + 2 = 0$
37. $2 \sin^2 x + 3 \sin x - 2 = 0$
38. $2 \cos^2 x - 5 \cos x - 3 = 0$

In Exercises 39–42, solve the equations on the interval $[0, 2\pi)$ and state the solutions to two decimal places.

39. $5 \sin^2 x - \sin x - 2 = 0$
40. $\sec^2 y - 5 \sec y + 6 = 0$
41. $3 \tan^2 u + 5 \tan u + 1 = 0$
42. $\cos^2 t - 2 \sin t + 3 = 0$

In Exercises 43–46, find the approximate solutions of the given equation in the interval $[0, 2\pi)$ by finding the point(s) of intersection of appropriate graphs on your graphing calculator.

43. $\cos x = x$
44. $\sin x = \cos x$
45. $\tan x = 8 - \frac{1}{2}x^2$
46. $3 \cos \frac{x}{2} = x^2 - 3$

Chapter Summary

Key Terms, Concepts, and Symbols

Key Ideas for Review

Topic	Page	Key Idea
Trigonometric Identity	465	A trigonometric identity is true for all values that may be assumed by the variable.
Fundamental Identities	466	The fundamental identities are trigonometric identities that are directly related to the definitions of the trigonometric functions.
Verification of Identities	467	The fundamental identities can be used to verify other trigonometric identities. The techniques frequently used to verify identities include: 1. factoring 2. writing trigonometric functions in terms of sine and cosine. 3. performing some of the indicated operations and simplifying complicated expressions 4. multiplying numerator and denominator of some fraction involving trigonometric functions by some trigonometric expression to obtain forms such as $1 - \sin^2 \theta$, $1 - \cos^2 \theta$, or $\sec^2 \theta - 1$, which can be further simplified.
Trigonometric Formulas	472	Some of the most useful trigonometric formulas are the following.
Addition Formulas	472	$\sin(s+t) = \sin s \cos t + \cos s \sin t$ $\cos(s+t) = \cos s \cos t - \sin s \sin t$ $\tan(s+t) = \dfrac{\tan s + \tan t}{1 - \tan s \tan t}$
Double-Angle Formulas	478	$\sin 2t = 2 \sin t \cos t$ $\cos 2t = \cos^2 t - \sin^2 t$ $\tan 2t = \dfrac{2 \tan t}{1 - \tan^2 t}$
Half-Angle Formulas	481	$\sin \dfrac{t}{2} = \pm\sqrt{\dfrac{1-\cos t}{2}}$ $\cos \dfrac{t}{2} = \pm\sqrt{\dfrac{1+\cos t}{2}}$ $\tan \dfrac{t}{2} = \pm\sqrt{\dfrac{1-\cos t}{1+\cos t}}$
Trigonometric Equations	488	Since the trigonometric functions are periodic, a trigonometric equation has either no solution or an infinite number of solutions.
Restricted Solutions	488	Sometimes the solutions to a trigonometric equation are restricted to a specific interval.

Review Exercises

In Exercises 1–3, verify the given identity.

1. $\sin\theta\sec\theta + \tan\theta = 2\tan\theta$
2. $\dfrac{\cos^2 x}{1-\sin x} = 1 + \sin x$
3. $\sin\alpha + \sin\alpha\cot^2\alpha = \csc\alpha$

In Exercises 4–7, determine the exact value of the given expression by using the addition formulas.

4. $\sin\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$
5. $\cos(45^\circ + 90^\circ)$
6. $\tan\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$
7. $\sin\frac{7\pi}{12}$

In Exercises 8–11, write the given expression in terms of cofunctions of complementary angles.

8. $\csc 15^\circ$
9. $\cos 23^\circ$
10. $\sin\frac{\pi}{8}$
11. $\tan\frac{2\pi}{7}$
12. If $\cos\theta = -\frac{12}{13}$ and $0 \le \theta \le 180^\circ$ find $\sin(\pi - \theta)$.
13. If $\sec\alpha = \frac{5}{4}$ and α lies in quadrant IV, find $\csc\left(\alpha + \frac{\pi}{3}\right)$.
14. If $\sin t = -\frac{3}{5}$ and $P(t)$ is in quadrant III, find $\tan(t + \pi)$.
15. If $\cos\alpha = -\frac{12}{13}$ and $\tan\beta = -\frac{5}{2}$, with angles α and β in quadrant II, find $\tan(\alpha + \beta)$.
16. If $\sin x = \frac{3}{5}$ and $\csc y = \frac{13}{12}$, with $P(x)$ in quadrant II and $P(y)$ in quadrant I, find $\cos(x - y)$.
17. If $\csc u = -\frac{5}{4}$ and $P(u)$ is in quadrant IV, find $\cos 2u$.
18. If $\tan\alpha = -\frac{3}{4}$ and $0 \le \alpha \le 180^\circ$ find $\sin 2\alpha$.
19. If $\sin 2t = \frac{3}{5}$ and $P(2t)$ is in quadrant I, find $\sin 4t$.
20. If $\sin\theta = 0.5$ and $\frac{\pi}{2} \le \theta \le \pi$ find $\sin 2\theta$.
21. If $\cos\frac{\theta}{2} = \frac{12}{13}$ and θ is acute, find $\sin\theta$.
22. If $\sin\alpha = -\frac{3}{5}$ and α is in quadrant III, find $\cos\frac{\alpha}{2}$.
23. If $\cot t = -\frac{4}{3}$ and $P(t)$ is in quadrant IV, find $\tan\frac{t}{2}$.
24. If $\cos 4x = \frac{2}{3}$ and $P(4x)$ is in quadrant IV, find $\cos 2x$.
25. Find the exact value of $\cos 15^\circ$ by using a half-angle formula.
26. Find the exact value of $\sin\frac{\pi}{8}$ by using a half-angle formula.
27. Find the exact value of $\tan 112.5^\circ$ by using a half-angle formula.

In Exercises 28–30, verify the given identity.

28. $\cos 30x = 1 - 2\sin^2 15x$
29. $\frac{1}{2}\sin 2y = \dfrac{\sin y}{\sec y}$
30. $\tan\frac{\alpha}{2} = \dfrac{1-\cos\alpha}{\sin\alpha}$
31. Express $\sin\frac{3\alpha}{2}\sin\frac{\alpha}{2}$ as a sum or difference.
32. Express $\cos 3x - \cos x$ as a product.
33. Evaluate $\sin 75^\circ \sin 15^\circ$ by using a product-sum formula.
34. Evaluate $\cos\frac{3\pi}{4} + \cos\frac{\pi}{4}$ by using a product-sum formula.

In Exercises 35–37, find all solutions of the given equation in the interval $[0, 2\pi)$. Express the answers in radian measure.

35. $2\cos^2\alpha - 1 = 0$
36. $2\sin\theta\cos\theta = 0$
37. $\sin 2t - \sin t = 0$

In Exercises 38–40, find all solutions of the given equation. Express the answers in degree measure.

38. $\cos^2\alpha - 2\cos\alpha = 0$
39. $\tan 3x + 1 = 0$
40. $4\sin^2 2t = 3$

41. Find the approximate solutions of $x\sin x = 10 - x^2$ in the interval $[0, 2\pi)$ by finding the points of intersection of appropriate graphs on your graphing calculator.

Review Test

1. Verify the identity $4 - \tan^2 x = 5 - \sec^2 x$.

In Exercises 2 and 3, determine exact values of the given expressions by using the addition formulas.

2. $\cos(270^\circ + 30^\circ)$
3. $\tan\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$
4. Write $\sin 47^\circ$ in terms of its cofunction.
5. If $\cos\theta = \frac{4}{5}$ and θ lies in quadrant IV, find $\sin(\theta - \pi)$.
6. If $\sin x = -\frac{5}{13}$ and $\tan y = \frac{8}{3}$ with angles x and y in quadrant III, find $\tan(x - y)$.
7. If $\sin v = -\frac{12}{13}$ and $P(v)$ is in quadrant IV, find $\cos 2v$.
8. If $\cos 2\alpha = -\frac{4}{5}$ and 2α is in quadrant II, find $\cos 4\alpha$.
9. If $\csc\alpha = -2$ and α is in quadrant III, find $\cos\frac{\alpha}{2}$.
10. Find the exact value of $\tan 15^\circ$ by using a half-angle formula.
11. Verify the identity
$$\sin\frac{x}{4} = 2\sin\frac{x}{8}\cos\frac{x}{8}$$
12. Express $\sin 2x + \sin 3x$ as a product.
13. Express $\sin 150^\circ - \sin 30^\circ$ by using a product-sum formula.
14. Find all solutions of the equation $4\sin^2\alpha = 3$ in the interval $[0, 2\pi)$. Express the answers in radian measure.
15. Find all solutions of the equation $\sin^2\theta - \cos^2\theta = 0$ and express the answers in degree measure.